Recall, from class notes, that

$$\begin{aligned}\tilde{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\alpha} - \hat{\beta}X_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(\alpha - \beta X_i + u_i - \hat{\alpha} - \hat{\beta}X_i)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(-(\hat{\alpha}-\alpha) - (\hat{\beta}-\beta)X_i + u_i\Big)^2.\end{aligned}$$

Since $\hat{\alpha} = \bar{Y} - \bar{X}\hat{\beta} = \alpha + \bar{X}\beta + \bar{u} - \bar{X}\hat{\beta} = \alpha - (\hat{\beta}-\beta)\bar{X} + \bar{u}$, we have $\hat{\alpha} - \alpha = -(\hat{\beta}-\beta)\bar{X} + \bar{u}$. Then,

$$\begin{aligned}\tilde{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}\Big((\hat{\beta}-\beta)\bar{X} - \bar{u} - (\hat{\beta}-\beta)X_i + u_i\Big)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(-(\hat{\beta}-\beta)(X_i-\bar{X}) + (u_i-\bar{u})\Big)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\Big((\hat{\beta}-\beta)^2(X_i-\bar{X})^2 + (u_i-\bar{u})^2 - 2(\hat{\beta}-\beta)(X_i-\bar{X})(u_i-\bar{u})\Big) \\ &= (\hat{\beta}-\beta)^2\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2 - 2(\hat{\beta}-\beta)\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(u_i-\bar{u}).\end{aligned}$$

Since, from class notes, $\hat{\beta}-\beta = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(u_i-\bar{u})}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}$, we have that

$$2(\hat{\beta}-\beta)\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(u_i-\bar{u}) = 2(\hat{\beta}-\beta)^2\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

Then,

$$\tilde{\sigma}^2 = (\hat{\beta}-\beta)^2\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2 - 2(\hat{\beta}-\beta)^2\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

In class, we obtained that

$$E\left((\hat{\beta}-\beta)^2\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2|X_1,\cdots,X_n\right) = \frac{\sigma^2}{n}, \text{ and}$$

$$E\left(\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2|X_1,\cdots,X_n\right) = \sigma^2 - \frac{\sigma^2}{n}.$$

Therefore,

$$E(\tilde{\sigma}^2|X_1,\cdots,X_n) = \frac{\sigma^2}{n} + \sigma^2 - \frac{\sigma^2}{n} - 2\frac{\sigma^2}{n} = \sigma^2\left(\frac{n-2}{n}\right). \quad (1)$$

Hence, we define the unbiased estimator

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{n}{n-2}\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{n}{n-2}\tilde{\sigma}^2.$$

Then, using (1)

$$E\left(\hat{\sigma}^2|X_1,\cdots,X_n\right) = \frac{n}{n-2}E\left(\tilde{\sigma}^2|X_1,\cdots,X_n\right) = \frac{n}{n-2}\sigma^2\left(\frac{n-2}{n}\right) = \sigma^2.$$

Also, by the Law of Iterated Expectations,

$$E\left(E\left(\hat{\sigma}^2|X_1,\cdots,X_n\right)\right) = E\left(\hat{\sigma}^2\right) = \sigma^2.$$